Phys 410
Spring 2013
Lecture #26 Summary
29 March, 2013

We finished the discussion of the Kepler orbits in the two-body problem. The orbit is described by $r(\varphi) = \frac{c}{1+\epsilon \cos \varphi}$, where $c = \frac{\ell^2}{\mu\gamma}$ is a length scale and ϵ is an un-determined constant. When the un-determined constant $\epsilon > 1$, the denominator of $r(\varphi)$ has a zero for some angle φ, and the particle is off at infinity for that angle. This is an un-bounded orbit, like those with energy $E > 0$. When $\epsilon < 1$ the values of $r(\varphi)$ are finite for all φ, and the orbit is bounded, like those with $E < 0$. The fact that $r(\varphi + 2\pi) = r(\varphi)$ means that the orbit is closed and periodic.

The orbit for $\epsilon < 1$ is an ellipse and is described by $\frac{(x+d)^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a = \frac{c}{1-\epsilon^2}$ is the semi-major axis, $b = \frac{c}{\sqrt{1-\epsilon^2}}$ is the semi-minor axis, and $d = a\epsilon$ is the distance from the center of the ellipse to the focus. The ratio of semi-minor to semi-major axis lengths is $b/a = \sqrt{1-\epsilon^2}$, showing that ϵ is the ellipticity of the orbit. One can also derive Kepler's third law of planetary motion relating the orbital period τ and the semi-major axis as $\tau^2 = \frac{4\pi^2}{GM_{sun}} a^3$ for the case of a planet orbiting the sun (here one assumes that the mass of the planet is much smaller than that of the sun). Finally we calculated the total mechanical energy in the center of mass frame as $E = \frac{\gamma^2\mu}{2\ell^2}(\epsilon^2 - 1)$. This shows that orbits with $\epsilon > 1$ are un-bounded (and described by a hyperbola), and those with $\epsilon < 1$ are bounded. Orbits with $\epsilon = 1$ are parabolic.

We next started a discussion of scattering theory. In the simplest scattering experiment one has a particle or entity (the projectile) that is launched with a known energy and momentum into a target, the projectile interacts with particles in the target, and then comes out as the same particle but with a new energy and momentum. More generally, the particle could be absorbed by the target, or be transformed into one or more different particles upon exiting the target. We can measure the exiting angle of the particle using spherical coordinates, with the z-axis along the initial projectile direction and the angular coordinates θ, φ specifying the new direction. Examples of scattering experiments include Rutherford scattering and angle-resolved photoemission spectroscopy (ARPES).

The only quantity not controlled in a typical scattering experiment is the impact parameter b of the projectile with respect to the target particle. The impact parameter is the distance of closest approach to the target particle, assuming no forces of interaction cause the projectile to change from its initial direction. Because we cannot control the impact parameter, we have to perform many experiments in which all possible values of b are employed for the incident beam of projectiles. We then give a statistical description of the resulting scattering. With such a description, we can write the number of particles scattered N_{scatt} in terms of the number of particles incident N_{inc} as $N_{scatt} = N_{inc} n_{target} \sigma$, where n_{target} is the density of target particles projected into the two-dimensional plane ($n_{target} \sim 1/m^2$) and σ is defined as the scattering cross section of each particle. σ is often measured in units of 'barns', which is $10^{-28} m^2$. We can generalize the concept of cross section to any process, including capture ($\sigma_{capture}$), ionization ($\sigma_{ionization}$), fission ($\sigma_{fission}$), etc. This is done by using the definition $N_{scatt,x} = N_{inc} n_{target} \sigma_x$ for process "x".

Experiments start with a beam of projectile particles of identical structure and equal initial momenta and energy. The projectiles enter the target with all possible values of impact parameter. One then measures how many particles come out with angle of exit θ, φ and also the energy and momentum of the exiting particle. Our job is to identify the force of interaction between the projectile and target particles from the number of particles scattered through angle θ, φ, for all possible angles. We write the 'angle-resolved' scattering cross section as $N_{scatt}(into\ d\Omega\ around\ \theta, \varphi) = N_{inc} n_{target} \frac{d\sigma}{d\Omega}(\theta, \varphi) d\Omega$, where $\frac{d\sigma}{d\Omega}(\theta, \varphi)$ is called the differential scattering cross section. Note that the element of differential solid angle is $d\Omega = 2\pi \sin\theta\, d\theta d\varphi$. We expect that if this quantity is integrated over all possible exiting angles, we should recover the total scattering cross section for this process: $\sigma = \iint \frac{d\sigma}{d\Omega}(\theta, \varphi)\, d\Omega$. We shall assume that all scattering potentials are spherically symmetric, hence there will be no dependence on the φ coordinate.

To find $\frac{d\sigma}{d\Omega}(\theta, \varphi)$ we compare the area covered by the incident particles at impact parameters between b and $b + db$ (i.e. $d\sigma = 2\pi b db$) to the solid angle subtended by the exiting beam of particles (i.e. $d\Omega = 2\pi \sin\theta\, d\theta$) to arrive at $\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$. To find this, we need to calculate the trajectory of a projectile particle for every possible impact parameter. We then did the example of a point particle elastically scattering from a fixed hard sphere of radius R and found that $\frac{d\sigma}{d\Omega} = \frac{R^2}{4}$, which is independent of angle! The total scattering cross section is just $\sigma = \pi R^2$, which is just the cross-sectional area of the sphere.